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A modified H_2 algorithm for improved frequency response function and nonlinear parameter estimation

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Abstract

H_1, H_2 and H_v are the most commonly used algorithms for frequency response function estimation. H_1 and H_v can be defined for multiple input–multiple output systems, whereas the H_2 estimator requires that the number of inputs equals the number of outputs for a unique solution. Otherwise, a non-square matrix needs to be inverted leading to numerical problems. These numerical problems can affect the Nonlinear Identification through Feedback of the Outputs (NIFO) nonlinear parameter estimation algorithm in which nonlinearities are treated as internal feedback forces (or inputs), which can lead to more inputs than outputs. In this paper, a modified form of the H_2 algorithm is presented that enables accurate estimates using NIFO. The modification to the H_2 algorithm is the addition of a correlated output, based on the nature of the nonlinearity in the system being identified, in order to make the matrix to be inverted square. Analytical models with multiple nonlinearities are used to show that this modification to H_2 leads to accurate estimates that are robust to noise on both the input and the output and that the modified algorithm is more robust to simultaneous noise on the input and output measurements than the H_1 and the H_v algorithms. Mathematical reasoning is used to explain the greater robustness of the modified H_2 algorithm over the traditional H_1 algorithm.

1. Introduction

Nonlinear identification through feedback of the outputs (NIFO) [1] is a nonlinear parameter estimation algorithm which identifies the underlying linear frequency response function (FRF) and the nonlinear parameters in one step by treating the nonlinearities as feedback forcing terms. Hickey et al. [2] used NIFO to identify the nonlinear behavior of vehicle suspension systems. Once the system nonlinearities have been characterized and the model has been formulated, a frequency response function estimation algorithm can be used to estimate the linear frequency response function and the nonlinear parameters.

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Various authors have examined the numerical issues associated with parameter estimation in linear system frequency response functions in the presence of noise on the input and output channels, see, for example Ref. [3]. Issues associated with these sources of noise in nonlinear system identification have also recently been examined by Josefsson et al. [4] who compared NIFO and the reverse-path method [5] for nonlinear parameter estimation with random noise signals using the H_1 algorithm. The comparison was performed with noise on the input; however, the H_1 algorithm minimizes the effects of noise on the output and not the input. This approach works well for the reverse-path method because it is formulated with the input as the output of the model. Consequently, Josefsson et al. showed that the reverse-path method provides more accurate estimates than NIFO when H_1 is used. Because the nonlinear outputs are treated as feedback forces in the NIFO formulation, any noise on the output also becomes part of the input. Consequently, the H_1 algorithm leads to biased estimates even if the measured input is not contaminated by noise. This issue can easily be overcome by using the H_v algorithm or H_2 if the noise is only on the measured input. However, the feedback of the outputs can lead to a system with unequal numbers of outputs and inputs. A necessary condition for a unique solution with the H_2 algorithm is that the number of inputs be equal to the number of outputs [3]; therefore, H_2 cannot be used in such cases.

This issue highlights an important aspect of nonlinear system identification; nonlinear identification methods cannot process noise in the same way as linear system identification methods. There is a greater bias in nonlinear identification due to measurement noise because even uncorrelated measurement noise becomes correlated due to the nonlinear elements in the system, for example, due to cubing operations as a result of nonlinear stiffness elements.

In this paper, a modification to the H_2 algorithm is presented that allows it to be used with NIFO for the case where the number of inputs is greater than the outputs. Correlated outputs, equal in number to the nonlinear feedback forces, are added to the set of system measurements such that the number of outputs equals the number of inputs and the matrix to be inverted is square. The choice of the correlated output terms is based on the form of the characterized nonlinearities in the system. It is shown that this modification to the H_2 algorithm not only gives reliable estimates for the case with noise on the input measurements but is also more robust than the H_1 and H_v algorithms in the case with simultaneous noise on the input and the output. Numerical simulations on a single-degree-of-freedom (sdof) Simulink® model are used to illustrate and compare the performance of the modified algorithm with both H_1 and H_v for different cases with noise on simulated data. The reason for the improved performance of the new H_2 algorithm compared to H_1 is explained mathematically.

The paper begins by reviewing the NIFO method and presenting the modified H_2 algorithm. The modified algorithm is then applied to nonlinear parameter estimation of a single-degree-of-freedom system with multiple nonlinearities and the robustness to noise is demonstrated.

2. Review of NIFO

In nonlinear systems, the external inputs act together with the internal nonlinear feedback forces on the underlying linear system to produce the measured outputs of the system. Fig. 1 illustrates the concept of internal feedback by the nonlinearities and superposition of the external forces and the internal feedback

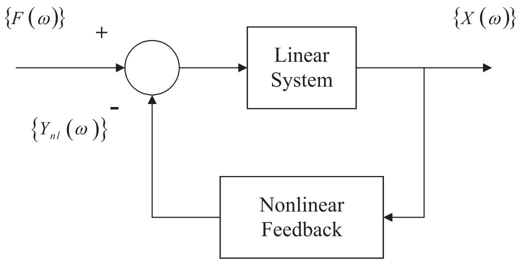

Fig. 1. Feedback by nonlinearities into a linear vibrating system illustrating the superposition of the external forces and the internal feedback forces.

forces. This feedback process is evident in the impedance model formulation for an arbitrary lumped parameter mechanical system given by

$$\mathbf{B}_L(\omega)\mathbf{X}(\omega) = \mathbf{F}(\omega) - \sum_{i=1}^{N} \mu_i(\omega)\mathbf{B}_{ni} Y_{\mathrm{nl}i}, \tag{1}$$

which can be written as

$$\mathbf{B}_L(\omega)\mathbf{X}(\omega) = \mathbf{F}(\omega) + \mathbf{F}_n(\omega), \tag{2}$$

where $\mathbf{B}_L(\omega)$ is the linear impedance matrix, $\mu_i(\omega)$ are the nonlinear (frequency dependent) coefficients, $\mathbf{F}(\omega)$ is the vector of Fourier transforms of the inputs, $\mathbf{X}(\omega)$ is the vector of Fourier transforms of the outputs, $Y_{\mathrm{nl}i}(\omega)$ are scalar Fourier transforms of the nonlinear restoring forces of the outputs, which account for the internal feedback forces, N is the number of nonlinearities included in the model, and $\mathbf{F}_n(\omega)$ is the vector of the nonlinear feedback forces. Each non-zero element of incidence vector, $\mathbf{B}_{ni}$, is either a 1 or $a{-}1$; these elements determine the location of the nonlinearity. A different $\mathbf{B}_{ni}$ and $Y_{\mathrm{nl}i}(\omega)$ pair is used to model each nonlinear element in the system.

The NIFO parameter estimation formulation is derived from Eq. (1) by multiplying both sides of the equation on the left by the linear system (square) frequency response function matrix, $\mathbf{H}_L(\omega)$, and separating the measured and unmeasured quantities as follows:

$$\mathbf{X}(\omega) = [\mathbf{H}_L(\omega)\ \ \mathbf{H}_L(\omega)\mu_1(\omega)\mathbf{B}_{n1}(\omega)\ \ \mathbf{H}_L(\omega)\mu_2(\omega)\mathbf{B}_{n2}(\omega)\cdots\mathbf{H}_L(\omega)\mu_N(\omega)\mathbf{B}_{N2}(\omega)]\begin{pmatrix} \mathbf{F}(\omega) \\ -\begin{pmatrix} Y_{\mathrm{nl}1}(\omega) \\ Y_{\mathrm{nl}2}(\omega) \\ \cdot \\ \cdot \\ \cdot \\ Y_{\mathrm{nl}N}(\omega) \end{pmatrix} \end{pmatrix}. \tag{3}$$

If the inputs and the outputs can be measured, and because the nonlinear functions can be calculated explicitly in terms of the measured inputs and the outputs, the set of equations in Eq. (3) can be used to estimate the best unbiased least squares estimate of the linear frequency response functions at the forced degrees of freedom and the nonlinear parameters $\mu_i(\omega)$ at forced and unforced degrees of freedom in a single step.

Consider a single-degree-of-freedom model. The equation of motion can be written in the frequency domain for zero initial conditions as follows when the nonlinear elements are taken to be frequency independent,

$$(-\omega^2 M + \mathrm{j}\omega C + K)X(\omega) + \mu Y_{\mathrm{nl}}(\omega) = F(\omega). \tag{4}$$

In this equation, $Y_{n1}(\omega)$ is the Fourier transform of a particular nonlinear feedback force, called a 'describing function', and μ is the corresponding nonlinear coefficient, assumed constant here, that determines the nonlinear weighting. This equation can be rewritten in the following form to highlight the presence of the nominal linear frequency response function:

$$X(\omega) = \frac{1}{(-\omega^2 M + \mathrm{j}\omega C + K)}F(\omega) - \frac{\mu}{(-\omega^2 M + \mathrm{j}\omega C + K)}Y_{\mathrm{nl}}(\omega). \tag{5}$$

Note that the coefficient of $F(\omega)$, $1/(-\omega^2 M + \mathrm{j}\omega C + K)$, is equal to the frequency response function of the underlying linear system (i.e., ratio of $X(\omega)$ to $F(\omega)$ when μ is zero); consequently, the nonlinear term in the second expression on the right-hand side of this equation determines the extent to which the system vibrates away from the linear operating point due to the presence of the nonlinearity.

If the linear frequency response function is replaced with the notation $H(\omega)$, Eq. (5) can be written in matrix form as follows:

$$X(\omega) = [H(\omega)\ \ \mu H(\omega)]\begin{bmatrix} F(\omega) \\ -Y_{\mathrm{nl}}(\omega) \end{bmatrix}. \tag{6}$$

Eq. (6) is equivalent to a two input, single output frequency domain model. Because the two inputs are not completely linearly correlated at any given frequency, a least squares parameter estimation procedure can be carried out to calculate the three coefficient functions in the row matrix on the right-hand side of the equation using spectral averaging via cross- and auto-power spectra. For example, if repeated runs using broad band stochastic (random) excitation are carried out, the results from each run can be assembled as

$$\{X(\omega)_1 \;\; X(\omega)_2 \;\; \cdots \;\; X(\omega)_{N_{\text{avg}}}\}_{1\times N_{\text{avg}}} = [H(\omega) \;\; \mu H(\omega)]_{1\times 2} \begin{bmatrix} F(\omega)_1 & F(\omega)_2 & \cdots & F(\omega)_{N_{\text{avg}}} \\ Y_{\text{nl}}(\omega)_1 & Y_{\text{nl}}(\omega)_2 & \cdots & Y_{\text{nl}}(\omega)_{N_{\text{avg}}} \end{bmatrix}_{2\times N_{\text{avg}}}, \tag{7}$$

where N_{avg} is the number of spectral averages. To solve this equation for the given number of spectral averages, the Hermitian transpose of the matrix on the right-hand side is multiplied on the right of both sides of the equation, thus producing cross- and auto-power matrices with the response data matrix and itself. These matrices are then amenable to standard cumulative spectral processing operations via H_1, H_2 and H_v calculations, whichever is more appropriate for the given assumptions regarding measurement noise. Note that unlike linear system identification, even uncorrelated response measurement noise can seriously corrupt the parameter estimates for nonlinear system identification because the nonlinear describing function (i.e., $Y_{n1}(\omega)$) always results in correlated noise.

3. The modified H_2 algorithm

The H_2 frequency response function estimator is given by the following equation:

$$\mathbf{H} = \mathbf{GXX}\,\mathbf{GFX}^{-1}, \tag{8}$$

where **GXX** is the auto-power matrix and **GFX** is the cross-power matrix. It is clear that the cross-power matrix **GFX** has to be square for a unique solution. This condition can only occur if the number of outputs equals the number of inputs. For the case where the nonlinear output of the single-degree-of-freedom system is considered as an internal feedback force, the following expression is derived:

$$\mathbf{GFX} = \begin{bmatrix} F_1 \\ -Y_{\text{nl}} \end{bmatrix} [X^*] = \begin{bmatrix} \text{GFX}_{11} \\ -\text{GYX}_{\text{nl1}} \end{bmatrix}. \tag{9}$$

This matrix is a 2×1, which is difficult to invert because of ill-conditioning. As an example, the H_2 algorithm was used to estimate the linear component and the nonlinear parameter of a single-degree-of-freedom system with cubic stiffness nonlinearity in Simulink® using NIFO (Eq. (6)). The parameters used in this simulation are given in Table 1. This problem involves two inputs and one output. A broad-band random input was used to obtain the estimates of the parameters using the H_2 algorithm as shown in Fig. 2. It is clear that both estimates are erroneous because a non-square matrix was inverted.

To circumvent the problem of inverting a 2×1 matrix, an extra correlated output, X_2, was added such that,

$$\mathbf{GFX} = \begin{bmatrix} F_1 \\ -Y_{\text{nl}} \end{bmatrix} [X_1^* \;\; X_2^*] = \begin{bmatrix} \text{GFX}_{11} & \text{GFX}_{12} \\ -\text{GYX}_{\text{nl1}} & -\text{GYX}_{\text{nl2}} \end{bmatrix}, \tag{10}$$

Table 1
Parameters of the single-degree-of-freedom nonlinear system with cubic stiffness nonlinearity

Mass (M)	Stiffness (K)	Damping (C)	Nonlinear parameter (μ)
0.0024	0.5	0.03	0.3

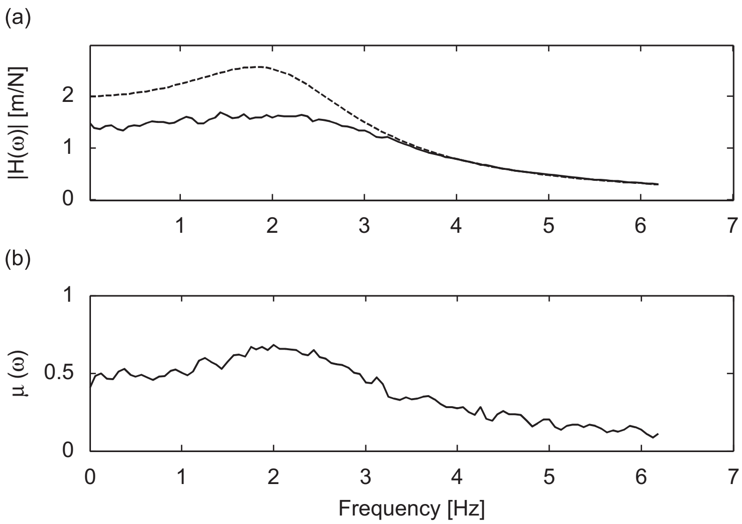

Fig. 2. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Estimates generated by using the Nonlinear Identification through Feedback of the Outputs and the original H_2 algorithm with an unequal number of inputs and outputs.

where $Y_{\mathrm{nl}} = X^3$ due to the cubic form of the nonlinearity. This matrix is size 2×2 and can be easily inverted. Substituting Eq. (10) into Eq. (8) with

$$\mathbf{GXX} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [X_1^* \;\; X_2^*] = \begin{bmatrix} \mathrm{GXX}_{11} & \mathrm{GXX}_{12} \\ \mathrm{GXX}_{21} & \mathrm{GXX}_{22} \end{bmatrix}, \tag{11}$$

gives

$$\mathbf{H} = \begin{bmatrix} \dfrac{(\mathrm{GXX}_{12})(\mathrm{GYX}_{\mathrm{nl1}}) - (\mathrm{GXX}_{11})(\mathrm{GYX}_{\mathrm{nl2}})}{\det(\mathbf{GFX})} & \dfrac{(\mathrm{GXX}_{12})(\mathrm{GFX}_{11}) - (\mathrm{GXX}_{11})(\mathrm{GFX}_{12})}{\det(\mathbf{GFX})} \\ \dfrac{(\mathrm{GXX}_{22})(\mathrm{GYX}_{\mathrm{nl1}}) - (\mathrm{GXX}_{21})(\mathrm{GYX}_{\mathrm{nl2}})}{\det(\mathbf{GFX})} & \dfrac{(\mathrm{GXX}_{22})(\mathrm{GFX}_{11}) - (\mathrm{GXX}_{21})(\mathrm{GFX}_{12})}{\det(\mathbf{GFX})} \end{bmatrix}, \tag{12}$$

where

$$\det(\mathbf{GFX}) = -(\mathrm{GFX}_{11})(\mathrm{GYX}_{\mathrm{nl2}}) + (\mathrm{GFX}_{12})(\mathrm{GYX}_{\mathrm{nl1}}). \tag{13}$$

If the extra output, $X_2 = X^3$ (nonlinear internal feedback force) is also considered to be an output, then Eq. (12) reduces to,

$$\mathbf{H} = \begin{bmatrix} \dfrac{(\mathrm{GXY}_{\mathrm{1nl}})(\mathrm{GYX}_{\mathrm{nl1}}) - (\mathrm{GXX}_{11})(\mathrm{GYY}_{\mathrm{nlnl}})}{\det(\mathbf{GFX})} & \dfrac{(\mathrm{GXY}_{\mathrm{1nl}})(\mathrm{GFX}_{11}) - (\mathrm{GXX}_{11})(\mathrm{GFY}_{\mathrm{1nl}})}{\det(\mathbf{GFX})} \\ \dfrac{\cancel{(\mathrm{GYY}_{\mathrm{nlnl}})(\mathrm{GYX}_{\mathrm{nl1}})} - \cancel{(\mathrm{GYX}_{\mathrm{nl1}})(\mathrm{GYY}_{\mathrm{nlnl}})}}{\det(\mathbf{GFX})} & \dfrac{(\mathrm{GYY}_{\mathrm{nlnl}})(\mathrm{GFX}_{11}) - (\mathrm{GYX}_{\mathrm{nl1}})(\mathrm{GFY}_{\mathrm{1nl}})}{-(\mathrm{GFX}_{11})(\mathrm{GYY}_{\mathrm{nlnl}}) + (\mathrm{GFY}_{\mathrm{1nl}})(\mathrm{GYX}_{\mathrm{nl1}})} \end{bmatrix}, \tag{14}$$

$$\Rightarrow \quad \mathbf{H} = \begin{bmatrix} H & \mu H \\ 0 & -1 \end{bmatrix}. \tag{15}$$

This algorithm was also used to estimate the quantities in Eq. (6) (Fig. 3). It is clear that the modification leads to accurate estimates of the linear frequency response function and the nonlinear parameter. The choice of the

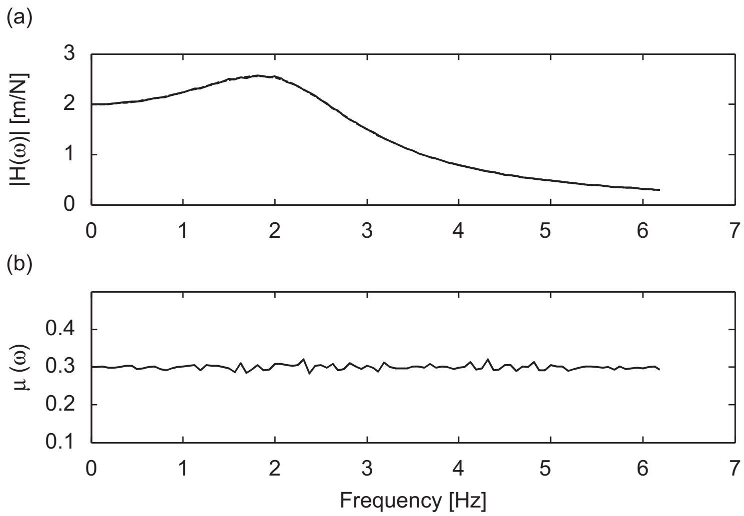

Fig. 3. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Estimates generated by using the Nonlinear Identification through Feedback of the Outputs and the modified H_2 algorithm with an equal number of inputs and outputs.

additional output is logical as it is related to the form of the nonlinear function and is actually a part of the actual output of the system. This choice also simplifies the second row of the resulting parameter matrix.

The modified H_2 algorithm was also applied to the case with two nonlinearities in the single-degree-of-freedom system. Quadratic and cubic stiffness nonlinearities were included in the model. The NIFO model for this system is

$$X(\omega) = [H(\omega)\ \ \mu_1 H(\omega)\ \ \mu_2 H(\omega)] \begin{bmatrix} F(\omega) \\ -X^2(\omega) \\ -X^3(\omega) \end{bmatrix}, \tag{16}$$

with $\mu_1 = 0.15$ and $\mu_2 = 0.3$.

First, the original H_2 algorithm was used to estimate the parameters in the model. Fig. 4 shows that as expected the estimates are erroneous. Next, the modified H_2 algorithm was used with both the nonlinear describing functions also used as outputs of the system. As in the previous case, the new algorithm provides accurate estimates of the parameters (Fig. 5).

The parameter matrix for this case is

$$\mathbf{H} = \begin{bmatrix} H & \mu_1 H & \mu_2 H \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{17}$$

The H_2 algorithm was also applied to nonlinear system identification of a two degree of freedom with similar improvements in the linear frequency response function and nonlinear parameter estimates with the modified H_2 algorithm.

3.1. Effect of choice of correlated output term on H_2 estimate

In the modified H_2 algorithm the form of the nonlinear function is chosen as the additional correlated output because it is a component of the output of the system. The results demonstrated that this choice is effective in parameter estimation. A range of power polynomials with orders between 0 and 30 were used to study the effect of the choice of the form of the correlated output on the H_2 algorithm. Fig. 6 shows the mean square error (MSE) in the linear frequency response function estimate and the absolute error in the nonlinear

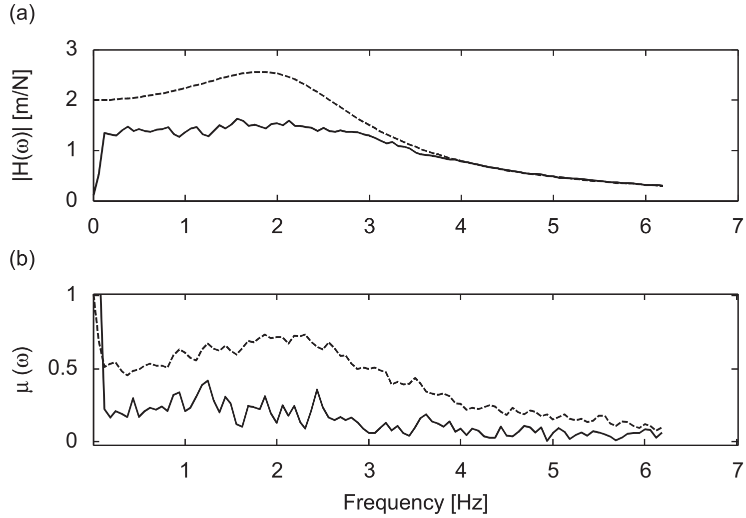

Fig. 4. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameters: μ_1 (—), μ_2 (- - - -). Estimates generated by using the Nonlinear Identification through Feedback of the Outputs and the original H_2 algorithm.

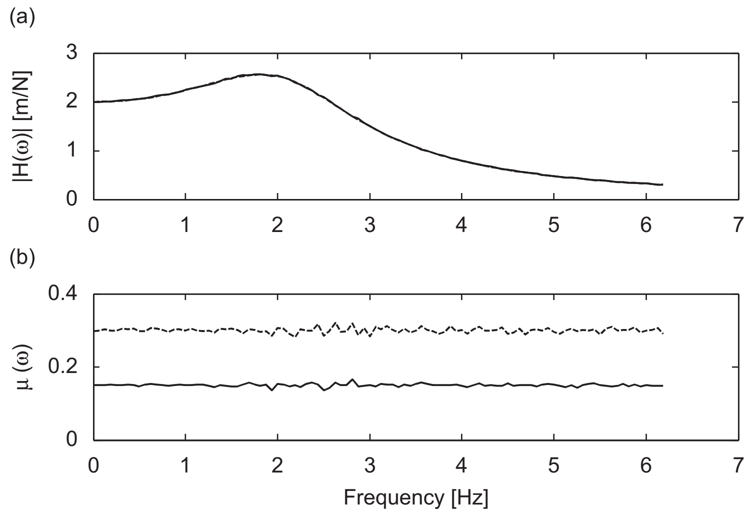

Fig. 5. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameters: μ_1 (—), μ_2 (- - - -). Estimates generated by using the Nonlinear Identification through Feedback of the Outputs and the modified H_2 algorithm.

parameter estimate, for cubic stiffness nonlinearity, as a function of the choice of the power of the additional output term, X^n. First, as expected, the power 1 gives the maximum error because the matrix to be inverted, **GFX**, becomes singular. $n = 3$ gives the best estimate because it is the same as the nonlinear function in the system. It is interesting to note that the linear frequency response function estimates are better for the odd powers as opposed to the even powers, and as the power increases the error fluctuation decreases and the mean error increases linearly. The reason for the lower errors for odd powers of X is that the form of the nonlinearity is odd and odd powers are better correlated with the output of the system. For powers beyond 5, the errors in the nonlinear parameter are actually lower for the even powers and higher for the odd powers. The nonlinear parameter is estimated by averaging over a range of values around the resonance of the system, which is possibly responsible for this apparent anomaly. It should also be noted that the amount of fluctuation

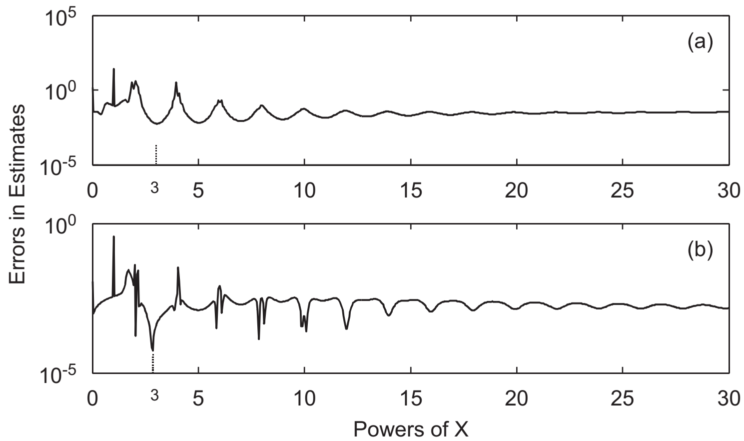

Fig. 6. (a) Mean square error in linear frequency response function magnitude estimate. (b) Absolute error in nonlinear parameter estimate. Effect of power of the nonlinear function (X^n) used as additional output on the estimates in the presence of a cubic nonlinearity, using the modified H_2 algorithm.

of the nonlinear parameter error values is 0.1–1%, which is quite small. The same analysis for quadratic stiffness nonlinearity revealed the same trend. The error in the linear FRF estimate fluctuates between low values for even powers and high values for odd powers and continually increases with the power of the nonlinear function of X.

4. Comparison of the modified algorithm with H_1 and H_v in the presence of noise

The H_1 algorithm minimizes noise on the output and the H_v algorithm is unaffected by noise on both the input and the output. H_1 is formulated as follows:

$$\mathbf{H} = \mathbf{GXF}\,\mathbf{GFF}^{-1}. \tag{18}$$

The H_v algorithm estimates the frequency response functions using the eigenvalue decomposition of a matrix of power spectrums [6]. According to one of the two formulations of H_v, for the current two input–one output case the following matrix of auto and cross powers can be defined:

$$\mathbf{GXFF} = \begin{bmatrix} \mathrm{GXX}_{11} & \mathrm{GXF}_{11} & \mathrm{GXY}_{1\mathrm{nl}} \\ \mathrm{GFX}_{11} & \mathrm{GFF}_{11} & \mathrm{GFY}_{1\mathrm{nl}} \\ \mathrm{GYX}_{\mathrm{nl}1} & \mathrm{GYF}_{\mathrm{nl}1} & \mathrm{GYY}_{\mathrm{nlnl}} \end{bmatrix}. \tag{19}$$

The eigenvalue decomposition is as follows:

$$\mathbf{GXFF} = \mathbf{V\Lambda V^H}, \tag{20}$$

where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues. The frequency response function and the nonlinear parameter can be found from the normalized eigenvector associated with the smallest eigenvalue (λ) as follows:

$$V_{\lambda_{\min}} = \begin{Bmatrix} -1 \\ H \\ \mu H \end{Bmatrix}. \tag{21}$$

As mentioned earlier, in nonlinear systems even uncorrelated measurement noise leads to bias errors in the estimates of system parameters. Hence, for example, the H_1 algorithm will not completely remove the effects of noise on the output for a nonlinear system. This result is especially true in the case of the NIFO formulation where the output is treated as a feedback force and as such the noise on the output becomes correlated noise on the input. The modified H_2 algorithm was compared with H_1 and H_v for three different cases of various levels of uncorrelated noise: (1) noise on the input; (2) noise on the output; and (3) noise on both input and output.

4.1. Single-degree-of-freedom system with cubic nonlinearity

NIFO was used for parameter estimation of a Duffing oscillator with the system parameters listed in Table 1. H_1, H_v and modified H_2 were used to solve Eq. (6). Rms noise of 0–50% was added to the input, output and then both the input and output in increments of 5%. The mean square error in the linear frequency response function and the absolute error in the nonlinear parameter estimate were calculated for each of the algorithms as a function of the noise for the three different cases.

4.1.1. Noise on the input

Fig. 7 shows the errors in the estimates with the three algorithms for the case of noise on the input, and Figs. 8–10 show the estimates of the linear frequency response function and the nonlinear parameter for 50% rms noise on the input using H_1, modified H_2 and H_v, respectively. As expected, H_2 provides accurate estimates in this case (Fig. 9). The estimates actually appear qualitatively more accurate than the H_v estimates (Fig. 10), and this is supported by the quantitative evidence in Fig. 7. H_1 provides the least accurate estimate of

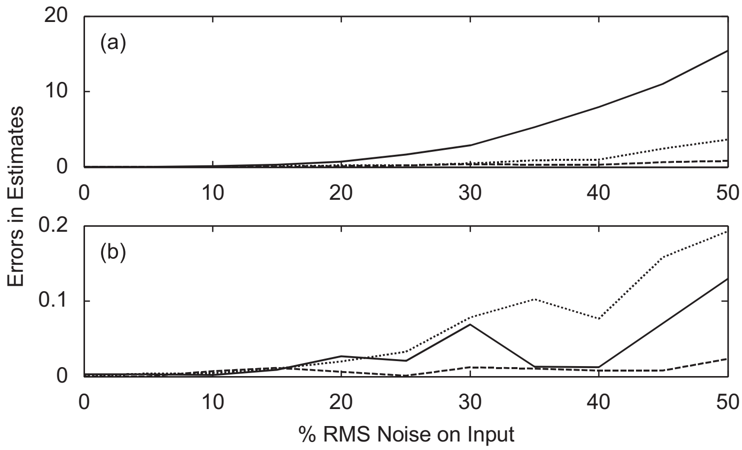

Fig. 7. Errors in the (a) magnitude of the linear frequency response function, and (b) the nonlinear parameter estimates, with noise on the input for the H_1 (—), modified H_2 (- - - -) and H_v (●●●●) algorithms.

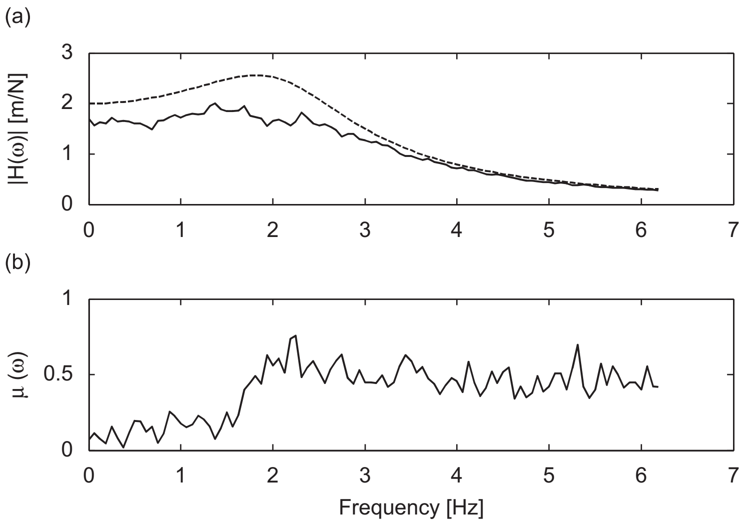

Fig. 8. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the H_1 algorithm with 50% rms noise on the input.

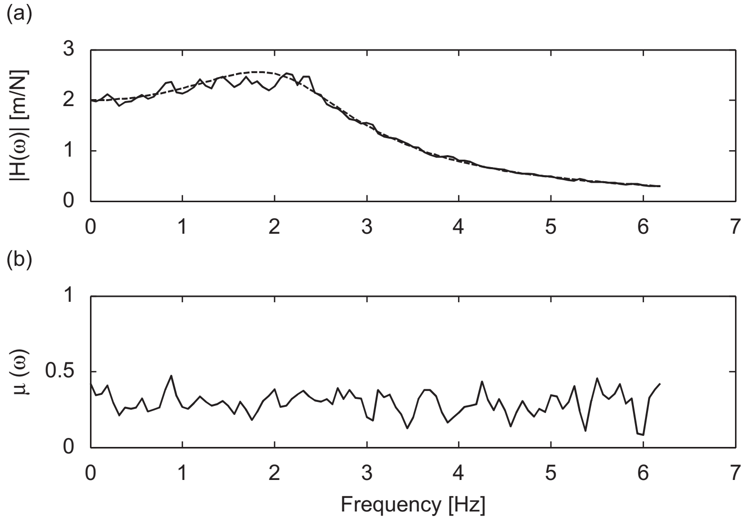

Fig. 9. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the modified H_2 algorithm with 50% rms noise on the input.

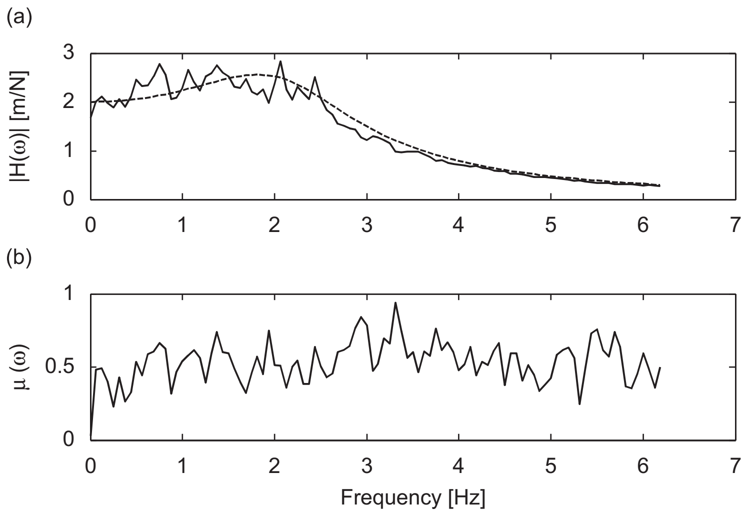

Fig. 10. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the H_v algorithm with 50% rms noise on the input.

the linear frequency response function but the nonlinear parameter estimates are better than for the H_v algorithm. As Fig. 8 shows, H_1 underestimates the linear frequency response function. The reason for this result can be explained by solving Eq. (18) for this case with noise on the input:

$$\mathbf{GXF}=[\mathrm{X}]\left[\left(F+\upsilon\right)^{*} \quad -Y_{\mathrm{nl}}^{\;*}\right]=\left[\mathrm{GXF}+\cancel{\mathrm{GX}\upsilon} \quad -\mathrm{GXY}_{\mathrm{nl}}\right], \tag{22}$$

where v is uncorrelated noise on the input, and

$$\mathbf{GFF}=\begin{bmatrix} F+\upsilon \\ -Y_{\mathrm{nl}} \end{bmatrix}[(F+\upsilon)^{*} \quad -Y_{\mathrm{nl}}^{*}]=\begin{bmatrix} \mathrm{GFF}+G\upsilon\upsilon & -\mathrm{GFY}_{\mathrm{nl}} \\ -\mathrm{GYF}_{\mathrm{nl}} & \mathrm{GYY}_{\mathrm{nl}} \end{bmatrix}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (18) gives

$$\mathbf{H} = \left[\frac{(\mathrm{GXF})(\mathrm{GYY_{nl}}) - (\mathrm{GXY_{nl}})(\mathrm{GYF_{nl}})}{\det(\mathbf{GFF})} \quad \frac{(\mathrm{GXF})(\mathrm{GFY_{nl}}) - \mathrm{GXY_{nl}}(\mathrm{GFF} + G\upsilon\upsilon)}{\det(\mathbf{GFF})} \right] = [H \; \mu H], \tag{24}$$

where

$$\det(\mathbf{GFF}) = (\mathrm{GFF} + G\upsilon\upsilon)\mathrm{GYY_{nl}} - (\mathrm{GFY_{nl}})(\mathrm{GYF_{nl}}). \tag{25}$$

Eqs. (24) and (25) explain the results in Fig. 8. The noise auto-power term *Gvv* in the denominator of Eq. (24) causes the linear frequency response function to be under-estimated, and this bias error becomes progressively worse as the noise level increases. The nonlinear parameter, on the other hand, has this noise term in both the numerator and the denominator. The denominator term dominates below resonance (around 2 Hz) and the numerator term dominates above resonance, leading to the estimate seen in Fig. 8.

The estimates with the modified H_2 algorithm can be obtained by solving Eq. (8), which results in

$$\mathbf{H} = \begin{bmatrix} \dfrac{(\mathrm{GXY_{nl}})(\mathrm{GYX_{nl}}) - (\mathrm{GXX})(\mathrm{GYY_{nl}})}{\det(\mathbf{GFX})} & \dfrac{(\mathrm{GXY_{nl}})(\mathrm{GFX}) - (\mathrm{GXX})(\mathrm{GFY_{nl}})}{\det(\mathbf{GFX})} \\ 0 & -1 \end{bmatrix}, \tag{26}$$

where

$$\det(\mathbf{GFX}) = -(\mathrm{GFX})(\mathrm{GYY_{nl}}) + (\mathrm{GFY_{nl}})(\mathrm{GYX_{nl}}). \tag{27}$$

It is clear that the noise does not affect the modified H_2 estimate, theoretically. The observed errors in Fig. 9 are due to the fact that it is difficult to numerically generate completely uncorrelated signals.

4.1.2. Noise on the output

Fig. 11 shows the errors in the estimates obtained from the three algorithms for the case of noise on the output, and Figs. 12–14 show the estimates of the linear frequency response function and the nonlinear parameter for 50% rms noise for the three algorithms. H_1 provides the most accurate estimates in this case (Fig. 12), and the modified H_2 algorithm is comparable to H_v (Figs. 11, 13 and 14), although it is not nearly as accurate as H_1. Solving Eq. (18) for this case with noise on the output gives

$$\begin{aligned} \mathbf{GXF} &= [X + \mu][F^* \quad - Y^*_{\mathrm{nl}}] \\ &= [X + \mu][F^* \quad - ((X + \mu)^3)^*] \\ &= [X + \mu][F^* \quad - X^*_{\mathrm{cn}}], \end{aligned} \tag{28}$$

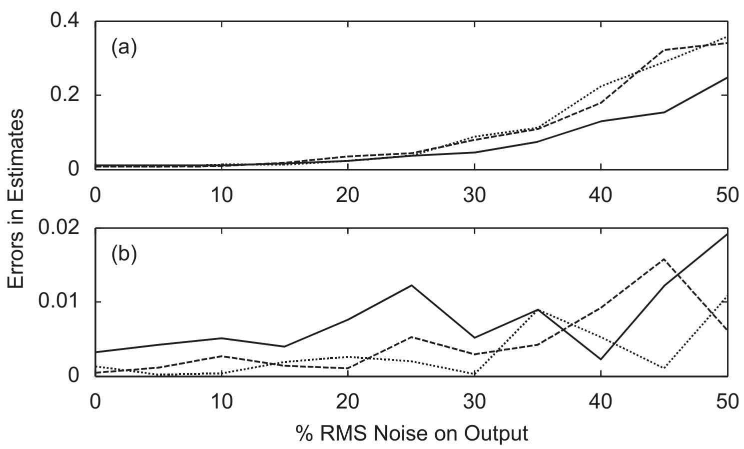

Fig. 11. Errors in the (a) magnitude of the linear frequency response function, and (b) the nonlinear parameter estimates, with noise on the output for the H_1 (—), modified H_2 (- - - -) and H_v (●●●●) algorithms.

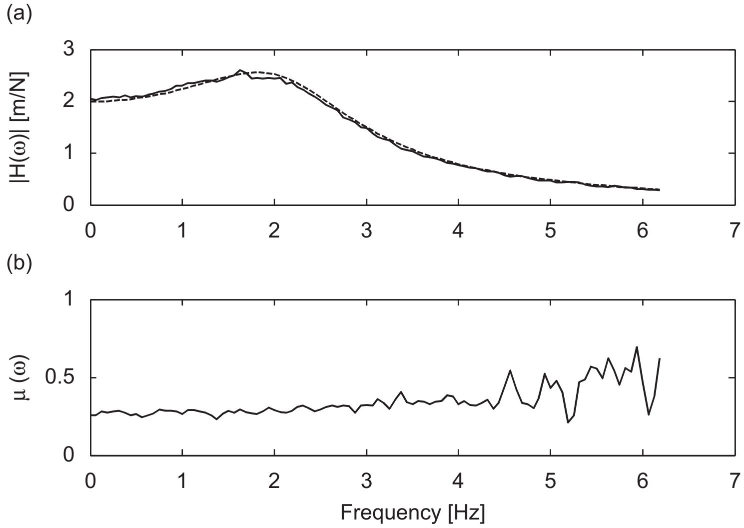

Fig. 12. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the H_1 algorithm with 50% rms noise on the output.

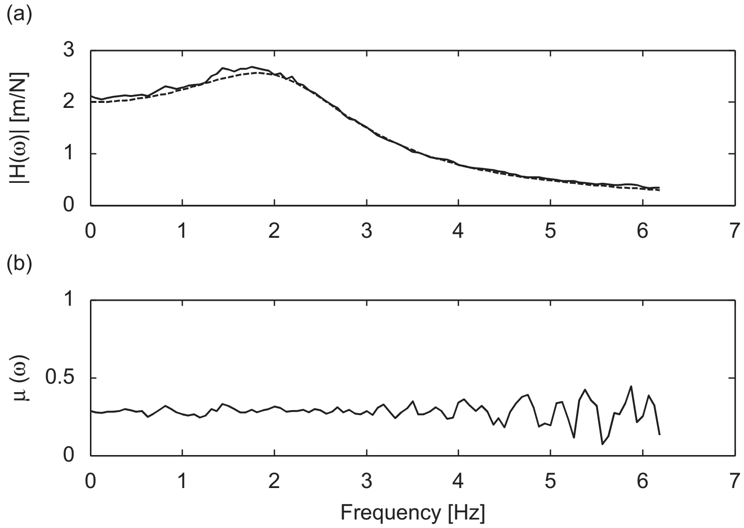

Fig. 13. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the modified H_2 algorithm with 50% rms noise on the output.

where μ is uncorrelated noise on the output. The feedback forcing term causes the noise on the output to also be part of the input in NIFO. It is treated as an output correlated noise term, X_{cn}. The resulting estimates are

$$\mathbf{H} = \left[\frac{(\mathrm{GXF})(\mathrm{GXX}_{\mathrm{cncn}}) - (\mathrm{GXX}_{\mathrm{cn}})(\mathrm{GXF}_{\mathrm{cn}}) - (G\mu X_{\mathrm{cn}})(\mathrm{GXF}_{\mathrm{cn}})}{\det(\mathbf{GFF})} \quad \frac{(\mathrm{GXF})(\mathrm{GFX}_{\mathrm{cn}}) - \mathrm{GFF}(\mathrm{GXX}_{\mathrm{cn}} + G\mu X_{\mathrm{cn}})}{\det(\mathbf{GFF})} \right]$$
$$= [H \ \ \mu H], \tag{29}$$

where

$$\det(\mathbf{GFF}) = (\mathrm{GFF})(\mathrm{GXX}_{\mathrm{cncn}}) - (\mathrm{GFX}_{\mathrm{cn}})(\mathrm{GXF}_{\mathrm{cn}}). \tag{30}$$

It is clear that the nonlinear feedback results in noise that contaminates the H_1 estimates.

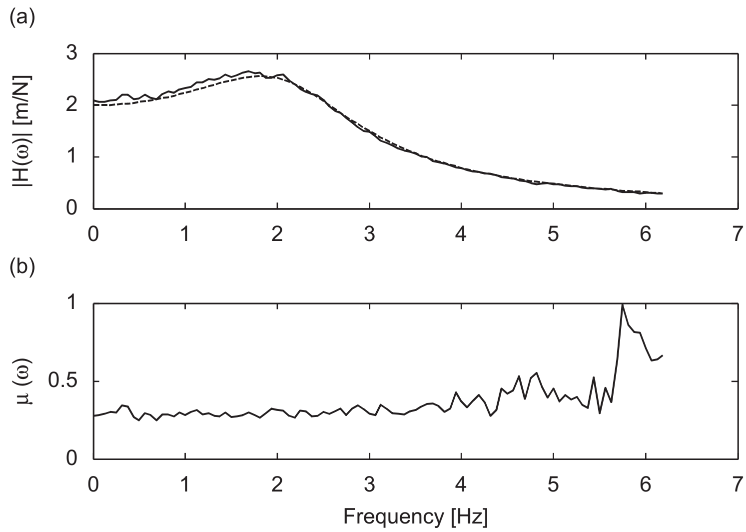

Fig. 14. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the H_v algorithm with 50% rms noise on the output.

The estimates with the modified H_2 algorithm can be obtained by solving Eq. (8), which results in

$$\mathbf{H}=\begin{bmatrix}\dfrac{\begin{array}{c}|\mathrm{GXX_{cn}}|+|\mathrm{GX}\mu_{\mathrm{cn}}|-(\mathrm{GXX})(\mathrm{GXX_{cncn}})\\-(G\mu\mu)(\mathrm{GXX_{cncn}})+(\mathrm{GXX_{cn}})(\mathrm{GX}\mu_{\mathrm{cn}})\\+(G\mu X_{\mathrm{cn}})(\mathrm{GXX_{cn}})\end{array}}{\det(\mathrm{GFX})} & \dfrac{\begin{array}{c}(\mathrm{GXX_{cn}}+G\mu X_{\mathrm{cn}})(\mathrm{GFX})\\-(G\mu\mu+\mathrm{GXX})(\mathrm{GFX_{cn}})\end{array}}{\det(\mathrm{GFX})}\\ 0 & -1\end{bmatrix}, \tag{31}$$

where

$$\det(\mathrm{GFX})=-(\mathrm{GFX})(\mathrm{GXX_{cncn}})+\mathrm{GFX_{cn}}(\mathrm{GXX_{cn}}+\mathrm{GX}\mu_{\mathrm{cn}}). \tag{32}$$

The modified H_2 algorithm is affected more by the noise and, hence, the estimates degrade slightly more with noise as compared to H_1. Despite this, the estimates are comparable in accuracy to those obtained with the H_v algorithm (Figs. 13 and 14).

4.1.3. *Noise on input and output*

Fig. 15 shows the errors in the estimates with the three algorithms for the case of simultaneous noise on the input and output, and Figs. 16–18 show the estimates of the linear frequency response function and the nonlinear parameter for 50% rms noise for the three algorithms. The modified H_2 algorithm provides the most accurate estimates in this case (Fig. 17), and the modified H_1 algorithm provides the least accurate estimates (Fig. 16). Solving Eq. (18) for the H_1 estimates gives

$$\mathbf{H}=\begin{bmatrix}\dfrac{(\mathrm{GXF})(\mathrm{GXX_{cncn}})-(\mathrm{GXX_{cn}})[\mathrm{GXF_{cn}}]-(G\mu X_{\mathrm{cn}})(\mathrm{GXF_{cn}})}{\det(\mathbf{GFF})} & \dfrac{\begin{array}{c}(\mathrm{GXF})(\mathrm{GFX_{cn}})-\mathrm{GFF}(GXX_{\mathrm{cn}}+G\mu X_{\mathrm{cn}})\\-G\upsilon\upsilon(\mathrm{GXX_{cn}}+G\mu X_{\mathrm{cn}})\end{array}}{\det(\mathbf{GFF})}\end{bmatrix}$$
$$=[\mathrm{H}\ \ \mu\mathrm{H}], \tag{33}$$

where

$$\det(\mathbf{GFF})=(\mathrm{GFF}+G\upsilon\upsilon)\mathrm{GXX_{cncn}}-(\mathrm{GFX_{cn}})(\mathrm{GXF_{cn}}). \tag{34}$$

As with the noise on input case, Eqs. (33) and (34) show that the noise auto-power term Gvv in the denominator of Eq. (33) causes the linear frequency response function to be under-estimated (Fig. 16), and

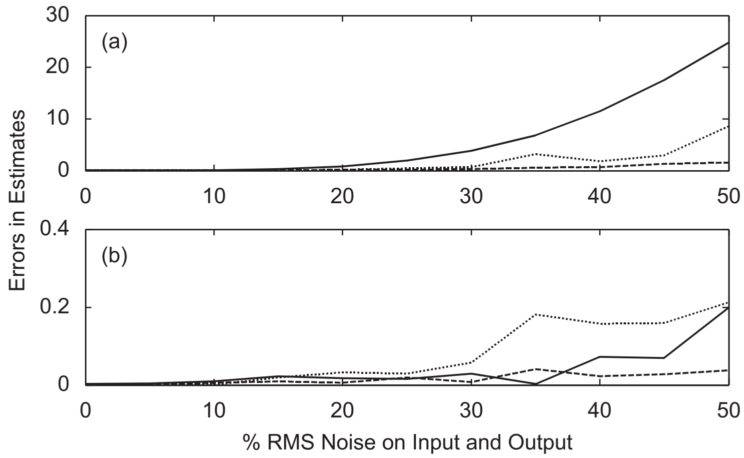

Fig. 15. Errors in the (a) magnitude of the linear frequency response function, and (b) the nonlinear parameter estimates, with noise on both the input and the output for the H_1 (—), modified H_2 (- - - -) and H_v (●●●●) algorithms.

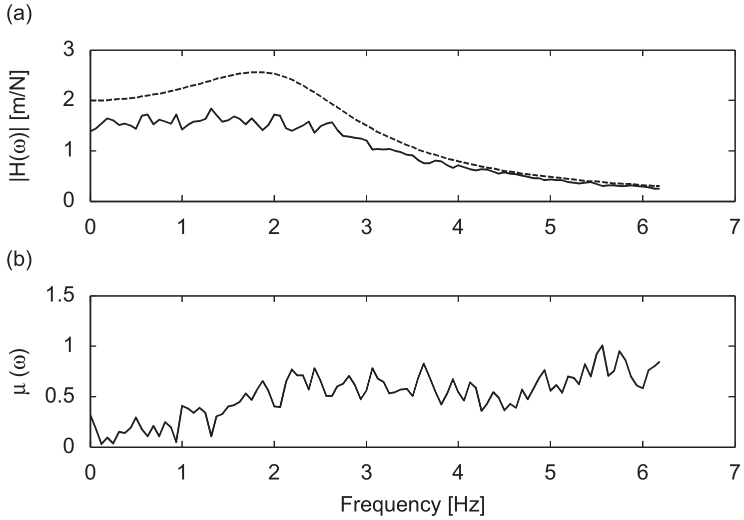

Fig. 16. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the H_1 algorithm with 50% rms noise on both the input and the output.

this error becomes progressively worse as the noise level increases. The nonlinear parameter, on the other hand, has this term in both the numerator and the denominator, along with output noise terms in the numerator. As with the input noise case, the denominator term dominates below resonance (around 2 Hz) and the numerator term dominates above resonance with less accurate estimates than the input noise case in the higher frequency range.

The estimates with the modified H_2 algorithm are the same as those for the case with noise on the output, Eq. (31). The estimates are more accurate than those obtained using either the H_1 or H_v algorithms in the presence of noise.

The analysis in this section shows that the modified H_2 algorithm is quite robust to noise. It provides better performance in terms of accuracy and robustness to noise of the estimates than H_1 except when noise is on the output, and even in that case it is only slightly less accurate in the presence of noise than H_1, which is expected to be the best estimator. In addition, the performance of the modified H_2 algorithm is either comparable to or better than the H_v algorithm in terms of accuracy and robustness to noise for all three noise cases.

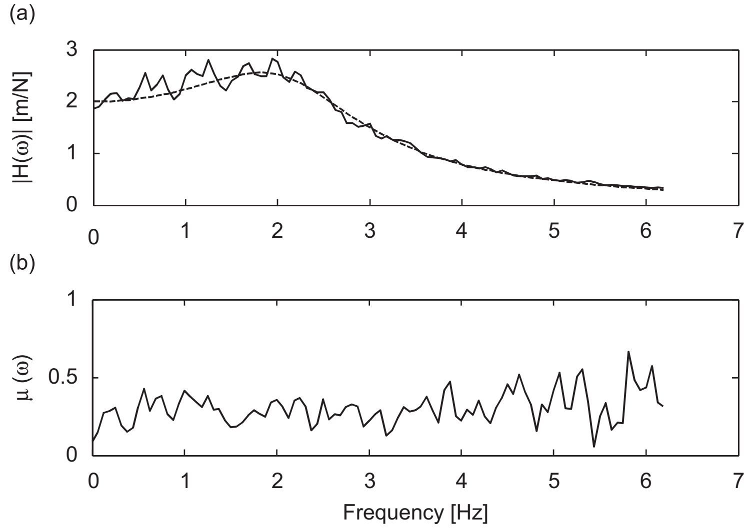

Fig. 17. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the modified H_2 algorithm with 50% rms noise on both the input and the output.

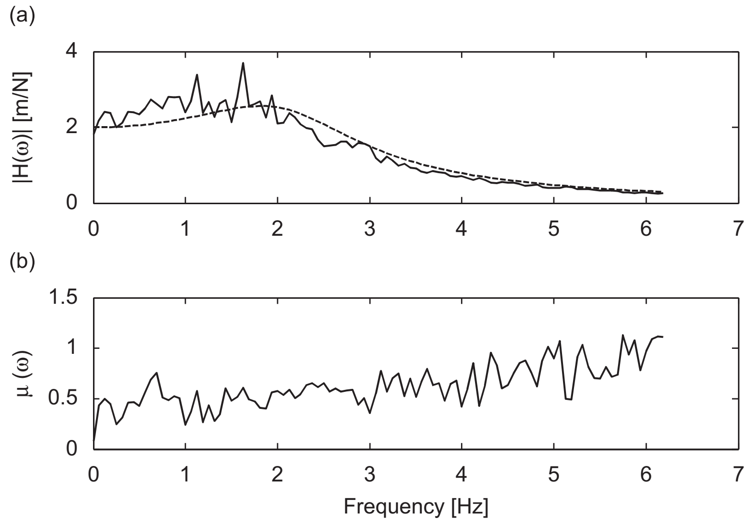

Fig. 18. (a) Magnitude of the linear frequency response function: actual (- - - -) and estimated (—). (b) The estimated nonlinear parameter. Nonlinear Identification through Feedback of the Outputs and the H_v algorithm with 50% rms noise on both the input and the output.

4.2. Single-degree-of-freedom system with quadratic and cubic nonlinearities

The same analysis as in the previous section was performed on the single-degree-of-freedom system with two nonlinearities, quadratic ($\mu_1 = 0.15$) and cubic ($\mu_2 = 0.3$). H_1, H_v and modified H_2 algorithms were used to solve Eq. (16). The relative performance of the three algorithms was found to be similar to the relative performance demonstrated in the previous analysis.

As an example, the errors in the estimates with the three algorithms for the case of simultaneous noise on input and output are shown in Fig. 19. As with the case involving a single nonlinearity, the modified H_2 algorithm provides the most accurate estimates in the presence of noise.

The analysis in this section demonstrates that the modification to the H_2 algorithm by adding a correlated output not only solves the problem of having to invert a non-square matrix for the case

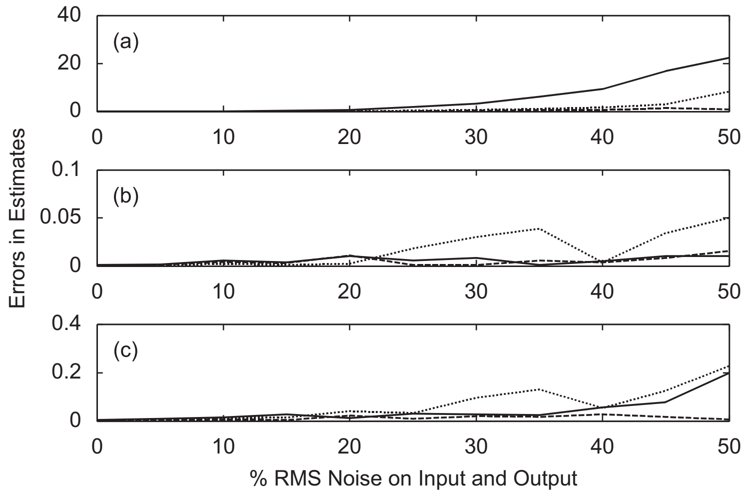

Fig. 19. Errors in the (a) magnitude of the linear frequency response function, (b) estimate of nonlinear parameter μ_1, and (c) estimate of nonlinear parameter μ_2, with noise on both the input and the output for the H_1 (——), modified H_2 (- - - -) and H_v (●●●●) algorithms; two nonlinearities.

of unequal inputs and outputs, but also makes the accuracy of parameter estimates quite robust to all noise sources.

5. Conclusions

A modified form of the H_2 frequency response function estimation algorithm was presented to overcome the issue of non-unique solutions when the number of inputs is greater than the number of outputs, which is often the case with nonlinear parameter estimation with feedback as in the Nonlinear Identification through Feedback of the Outputs method. Correlated outputs were added to the system, based on the form of the nonlinear function, to make the inverted matrix square. It was shown that the modified algorithm is robust to noise on the input and output and actually out-performs H_1 and H_v for the cases of noise on the input and simultaneous noise on both input and output. The greater robustness to noise on both the input and the output is important because the feedback nature of nonlinear parameter estimation algorithms results in noise on the output also becoming part of the input data. Mathematical derivations of the H_1 and modified H_2 algorithm were used to explain the greater robustness of the modified H_2 algorithm over the traditional H_1 algorithm.

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